

Integrability Conditions for Irrotational Dust with a Purely Electric Weyl Tensor: A Tetrad Analysis

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Abstract

All spacetimes for an irrotational collisionless fluid with a purely electric Weyl tensor, with spacetime curvature determined by the exact Einstein field equations, are shown to be integrable. These solutions include the relativistic generalisations of the Zeldovich solutions of Newtonian theory. Thus our result shows the consistency of various studies of "silent" universes (where such consistency was assumed rather than proved).

Subject headings:

cosmology - galaxies: clustering,
formation - hydrodynamics - relativity - exact solutions

1 Introduction

The dynamics of self-gravitating collisionless matter (dust) that is irrotational and with vanishing magnetic part of the Weyl tensor has been the focus of interest in the last couple of years [6, 7, 2].

The assumption that the matter can be described by a collisionless fluid is reasonable since we wish to study cosmological perturbations during the matter dominated era. However, because of the high degree of non-linearity in Einstein's field equations, a number of additional physically motivated approximations have been introduced to make the problem more tractable.

The second assumption is that the fluid flow is irrotational. Physically this is reasonably well justified provided the scales of interest are not too small (i.e. where rotation starts to become important). Then the kinematical condition that $\omega_{ab} = 0$ implies the flow is hypersurface orthogonal. Furthermore if it is initially irrotational then it will remain irrotational in the absence of dissipative effects. It is the third assumption of vanishing "magnetic" part of the Weyl tensor that has led to most discussion.

A remarkable feature of perfect fluid spacetimes with vanishing vorticity ω_{ab} and "magnetic" part of the Weyl tensor H_{ab} is that they admit an orthonormal tetrad associated with the matter 4-velocity u^a which is simultaneously an eigen-tetrad for the shear of the matter flow σ_{ab} and the "electric" part of the Weyl tensor E_{ab} [1]. It follows that, apart from two special cases, coordinates exist in which the metric g_{ab} and the tensors σ_{ab} and E_{ab} are all diagonal. This together with the further condition that the flow is geodesic $\dot{u}_a = 0$ leads to a considerable simplification of the propagation equations, reducing them to a set of six ordinary differential equations (the Raychaudhuri and continuity equations together with two pairs of equations for the independent components of σ_{ab} and E_{ab}) [6]. In addition to these propagation equations various constraint equations have to be satisfied that result from these conditions. Apart from the equations that directly express the restriction $H_{ab} = 0$, there is an additional constraint which follows from the \dot{H} Bianchi identity. Physically this can be thought of as limiting the spatial variation of the "electric" tidal field E_{ab} .

When these constraint equations are satisfied, each fluid element evolves independently of each other until the formation of caustics, when the one-to-one mapping between fluid elements and space points is lost. Such a universe model was dubbed *silent* by Matarrese *et. al.* [7], because no information can be exchanged between the fluid elements which was not already present in the initial conditions (which must be chosen to satisfy the constraint equations).

The aim of this paper is to show that the required integrability conditions are satisfied: the time-propagation equations are consistent with the constraints. Two of the constraints are clearly sensitive to the assumption of $H_{ij} = 0$ and warrant a full consistency check: *viz* i) the usual $H = 0 = w$ constraint which restricts the spatial distribution of the shear σ_{ij} ; ii) and the additional constraint due to $\dot{H}_{ij} = 0$ described above. In fact in showing that in the case of dust, vectors of the tetrad are hypersurface orthogonal, Barnes and Rowlingson (1989) [1] performed a consistency test on the part of the \dot{H}_{ij} constraint for which $i = j$. We carry out a full consistency on all the constraints.

Sections 3 and 4 list the evolution and the constraints equations in covariant form. Results of calculation of the time derivative of constraints are given in section 5, using a tetrad formalism. A sample calculation appears in the appendix. Consistency analysis is performed first for the trivial Type O fields, which evolve as Friedmann-Robertson-Walker (FRW) models. The type D and type I fields are tested separately. In both cases, integrability conditions are consistently satisfied.

Kinematic and tetrad notation used here are the same as in Ellis (1967) [3]. Latin indices run from 0 to 3 and Greek indices from 1 to 3. The tetrad $\{e_0, e_i\}$ is chosen with the timelike vector e_0 chosen as the fluid flow vector, and the associated parameter τ measuring proper time between the surfaces orthogonal to the fluid flow (spanned by the vectors e_i). This can be done since for dust we have zero acceleration. Such a specialization simplifies time propagation calculations.

2 Relativistic dynamics of irrotational dust

In this section we will give a brief summary of the relativistic dynamics of irrotational dust [3] in terms of covariant variables that represent the observable kinematical and dynamical quantities [4], focusing on the case of irrotational dust with vanishing “magnetic part” of the Weyl tensor: $p = \omega_{ab} = H_{ab} = 0$.

2.1 Variables

The relativistic dynamics of dust is determined by the Einstein Field equations and by the continuity equation for the matter stress-energy-tensor $T_{ab} = \rho u_a u_b$, where ρ is the energy density and u^a is the normalized 4-velocity of the fluid ($u^a u_a = -1$). At each spacetime point we can define a projection tensor $h_{ab} = g_{ab} + u_a u_b$ for which $h_{ab} u^a = 0$. With u^a and h_{ab} it is possible to split the covariant derivative of any tensorial quantity into a time derivative and a spatial derivative. In particular the first covariant derivative of u^a can be written as $u_{a;b} = v_{ab} + a_a u_b$, where $v_{ab} \equiv h^c_a h^d_b u_{c;d}$ is the spatial part satisfying $v_{ab} u^b = 0$, and $a^a \equiv \dot{u}^a \equiv u^a_{;b} u^b$ is the acceleration vector which is orthogonal to the fluid flow ($a^a u_a = 0$). It is standard to split v_{ab} into three kinematical quantities: its trace $\Theta \equiv v^a_{;a}$, symmetric, trace-free part $\sigma_{ab} \equiv v_{(ab)} - \frac{1}{3} h_{ab} \Theta$, and antisymmetric part $\omega_{ab} \equiv v_{[ab]}$. The physical meaning of these quantities are as follows: Θ measures the volume expansion or contraction of the fluid, σ_{ab} is the shear tensor which describes the rate at which a spherical fluid volume is distorted into an ellipse and ω_{ab} is the vorticity tensor giving the rate of rotation with respect to a local inertial frame.

2.2 Kinematic equations

For a perfect fluid the fluid acceleration is only determined by pressure gradients so the restriction of vanishing pressure implies that $a^a = 0$. This means that each fluid element moves along a geodesic. The conservation of energy and momentum $T^{ab}_{;b} = 0$ leads to only one further equation, the continuity equation:

$$\dot{\rho} = -\rho \Theta . \quad (1)$$

With the second restriction of vanishing vorticity $\omega_{ab} = 0$, the equations for the kinematic quantities follow from the Ricci identity: $u_{a;d;c} - u_{a;c;d} = R_{abcd}u^b$ [4,5]. The expansion scalar Θ obeys the Raychaudhuri equation:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + \frac{1}{2}\kappa\rho = 0 , \quad (2)$$

where $\sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}$ is the shear scalar and $\kappa = 8\pi G$ is the gravitational constant. Note that in the homogeneous and isotropic case where $\sigma_{ab} = 0$ and $\Theta = 3H$, this equation reduces to the familiar Friedmann equation:

$$3\dot{H} + 3H^2 + \frac{1}{2}\kappa\rho = 0 . \quad (3)$$

The only other kinematic equation is for the shear. It is given by:

$$\dot{\sigma}_{ab} + \sigma_{ac}\sigma^c_b - \frac{2}{3}\sigma^2 h_{ab} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} = 0 , \quad (4)$$

where $E_{ac} = E_{(ac)} \equiv C_{abcd}u^b u^d$ is the “electric” part of the Weyl tensor C_{abcd} (satisfying $E_{ac}u^c = 0$, $E^a_a = 0$). E_{ab} is that part of the gravitational field which describes tidal interactions. The Weyl tensor can be decomposed into E_{ab} and another tensor called the “magnetic” part: $H_{ac} = H_{(ac)} \equiv \frac{1}{2}\eta_{ab}{}^{gh}C_{ghcd}u^b u^d$ (satisfying $H_{ac}u^c = 0$, $H^a_a = 0$). This is the part of the gravitational field that describes magneto-gravitic effects, and allows gravitational waves.

So far we have explicitly made two assumptions, those of vanishing pressure and vorticity. In what follows we will make the third and most important assumption, namely, neglecting the influence of the “magnetic” part of the Weyl tensor. This implies neglecting the interaction of gravitational waves with the system.

2.3 Constraint equations

Besides the evolution equations for the kinematical equations, there are several constraints that our variables must satisfy. On setting $p = \omega_{ab} = H_{ab} = 0$ we obtain as non-trivial constraints,

$$h^e_b \left(\frac{2}{3}\Theta^{;b} - h^d_c \sigma^{bc}_{;d} \right) = 0 , \quad (5)$$

the ‘ $(0, \nu)$ ’ field equations, and

$$h^t_a h^s_d \sigma_{(t}{}^{b;c}{}_{s)} \eta_{fbc} u^f = 0 , \quad (6)$$

the condition that $H_{ab} = 0$. Additionally there is a Friedmann like equation giving the Ricci-scalar of the 3-spaces orthogonal to u^a ; this is a first-integral of (1) and (2).

2.4 Bianchi Identities

Additionally, the Bianchi identities must be satisfied, as they are the integrability conditions for the other equations. With our restrictions, they take the form:

$$h^t_a h^d_s E^{as}{}_{;d} = \frac{1}{3} h^t_b \rho^{;b}{}_{;t} , \quad (7)$$

$$h^m_a h^t_c \dot{E}^{ac} + h^{mt} \sigma^{ab} E_{ab} + \Theta E^{mt} - 3 E_s^{(m} \sigma^{t)s} = -\frac{1}{2} \rho \sigma^{tm} , \quad (8)$$

the ‘div E ’ and ‘ \dot{E} ’ equations respectively, and

$$\eta^{tbpq} u_b \sigma^d_p E_{qd} = 0 , \quad (9)$$

and

$$h_a^{(m} \eta^{t)rsd} u_r E^a_{s;d} = 0 . \quad (10)$$

which are the ‘div H ’ and ‘ \dot{H} ’ equations respectively in the case where H_{ab} vanishes.

3 Tetrad approach

A direct conversion of these equations into a tetrad system which is an eigen-frame for both the shear tensor and the Weyl tensor yields the following time-evolution equations (Barnes and Rowlingson, 1989 [1]);

$$\begin{aligned} \dot{\rho} &= -\rho\theta , \\ \dot{\theta} &= -\frac{1}{3}\theta^2 - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{2}\rho , \\ \dot{\sigma}_\mu &= -(\sigma_\mu)^2 - \frac{2}{3}\theta\sigma_\mu + \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - E_\mu , \\ \dot{E}_\mu &= -\theta E_\mu - \frac{1}{2}\rho\sigma_\mu + 3\sigma_\mu E_\mu - (\sigma_1 E_1 + \sigma_2 E_2 + \sigma_3 E_3) , \end{aligned} \quad (11)$$

being respectively the tetrad forms of (1), (2), and of the diagonal parts of (4) and (8). The non-diagonal parts of (4) and (8) (the propagation equations for \dot{E}_{ab} and $\dot{\sigma}_{ab}$ with $a \neq b$) introduces two additional conditions;

$$0 = (\sigma_\nu - \sigma_\mu)\Gamma^\nu_{0\mu}, \quad \mu \neq \nu, \quad (12)$$

$$0 = (E_\nu - E_\mu)\Gamma^\nu_{0\mu} \quad \mu \neq \nu, \quad (13)$$

where the Ricci rotation coefficients are defined by $\Gamma_{abc} := e_a \cdot \nabla_b e_c$ [3], and one can raise and lower indices in the first place as usual by use of the metric.

3.1 Uniqueness of tetrad

The tetrad used here is an orthonormal tetrad. If we denote the local coordinate system by $\{x^i\}$ and the tetrad by $\{\mathbf{e}_a\}$ then the equations (Ellis, 1967 [3])

$$\mathbf{e}_a = e_a^i (\partial/\partial x^i) \quad (14)$$

define the functions e_a^i , which are components of the tetrad vectors \mathbf{e}_a with respect to the basis $\partial/\partial x^i$, and are also directional derivatives of the coordinate functions x^i as

$$e_a^i = \partial_a(x^i). \quad (15)$$

The choice of a tetrad which simultaneously diagonalizes the shear tensor and the Weyl tensor possess the following uniqueness properties:

1. The tetrad is uniquely determined for the two cases:
 - (a) Distinct E_μ and/or distinct σ_μ values.
 - (b) Degenerate shear tensor and Weyl tensor, with the principal planes not coinciding i.e., (say) $E_1 = E_2 \neq E_3$ and $\sigma_1 = \sigma_3 \neq \sigma_2$. Here equations (12) and (13) yield

$$\Gamma^\nu_{0\mu} = 0; \quad \mu \neq \nu. \quad (16)$$

2. The tetrad is free by a rotation in (say) the e_1, e_2 plane if both the shear and the Weyl tensor are degenerate in the same plane i.e., $E_1 = E_2 \neq E_3$ and $\sigma_1 = \sigma_2 \neq \sigma_3$.

We write the tetrad freedom as

$$\begin{aligned}\bar{e}_1^i &= e_1^i \cos \phi + e_2^i \sin \phi , \\ \bar{e}_2^i &= -e_1^i \sin \phi + e_2^i \cos \phi ; , \quad \phi = \phi(x^0, x^i)\end{aligned}\quad (17)$$

and from (12) or (13) we have

$$\Gamma^3_{02} = \Gamma^3_{01} = 0 . \quad (18)$$

A rotation of the tetrad can then be performed to determine $\partial\phi/\partial x^0$ from the requirement that

$$\Gamma^2_{01} = 0 . \quad (19)$$

This leaves the tetrad arbitrary by an initial rotation in a surface $x^0 = \text{const.}$ The remaining freedom is later used to set $\Gamma^1_{32} = 0$ in (51) below.

Equation (16) is thus valid generally and simplify calculations greatly. Both equations (12) and (13) are satisfied. Also it follows trivially from the propagation equations that $E_\mu = 0$ if and only if $\sigma_\mu = 0$ (FRW model).

4 Constraint Equations

The constraint equations give spatial restrictions on the tetrad form of the dynamical variables. The $(0, \nu)$ field equations (5) are:

$$\frac{2}{3}\partial_1\theta = \partial_1\sigma_1 + (\sigma_1 - \sigma_2)\Gamma^2_{21} + (\sigma_1 - \sigma_3)\Gamma^3_{31} , \quad (20)$$

$$\frac{2}{3}\partial_2\theta = \partial_2\sigma_2 + (\sigma_2 - \sigma_1)\Gamma^1_{12} + (\sigma_2 - \sigma_3)\Gamma^3_{32} , \quad (21)$$

$$\frac{2}{3}\partial_3\theta = \partial_3\sigma_3 + (\sigma_3 - \sigma_1)\Gamma^1_{13} + (\sigma_3 - \sigma_2)\Gamma^2_{23} . \quad (22)$$

The ‘div E ’ equations (7) take the form

$$\frac{1}{3}\partial_1\rho = \partial_1E_1 + (E_1 - E_2)\Gamma^2_{21} + (E_1 - E_3)\Gamma^3_{31} , \quad (23)$$

$$\frac{1}{3}\partial_2\rho = \partial_2E_2 + (E_2 - E_1)\Gamma^1_{12} + (E_2 - E_3)\Gamma^3_{32} , \quad (24)$$

$$\frac{1}{3}\partial_3\rho = \partial_3E_3 + (E_3 - E_1)\Gamma^1_{13} + (E_3 - E_2)\Gamma^2_{23} . \quad (25)$$

The ‘div H ’ equations (9) are identically satisfied by the tetrad variables (indeed it is they that allowed us to simultaneously diagonalise the shear and

the Electric part of the Weyl tensor).

The equation (6) for vanishing H_{ab} takes the form

$$\Gamma^1_{32}(\sigma_2 - \sigma_1) = \Gamma^1_{23}(\sigma_3 - \sigma_1) , \quad (26)$$

$$\Gamma^2_{31}(\sigma_1 - \sigma_2) = \Gamma^2_{13}(\sigma_3 - \sigma_2) , \quad (27)$$

$$\Gamma^3_{12}(\sigma_2 - \sigma_3) = \Gamma^3_{21}(\sigma_1 - \sigma_3) , \quad (28)$$

$$\partial_1(\sigma_3 - \sigma_2) = \Gamma^3_{31}(\sigma_1 - \sigma_3) - \Gamma^2_{21}(\sigma_1 - \sigma_2) , \quad (29)$$

$$\partial_2(\sigma_3 - \sigma_1) = \Gamma^3_{32}(\sigma_2 - \sigma_3) - \Gamma^1_{12}(\sigma_2 - \sigma_1) , \quad (30)$$

$$\partial_3(\sigma_2 - \sigma_1) = \Gamma^2_{23}(\sigma_3 - \sigma_2) - \Gamma^1_{13}(\sigma_3 - \sigma_1) . \quad (31)$$

Finally the dynamical restriction $\dot{H}_{\mu\nu} = 0$, equation (10), introduces constraints given by

$$\Gamma^1_{32}(E_2 - E_1) = \Gamma^1_{23}(E_3 - E_1) , \quad (32)$$

$$\Gamma^2_{31}(E_1 - E_2) = \Gamma^2_{13}(E_3 - E_2) , \quad (33)$$

$$\Gamma^3_{12}(E_2 - E_3) = \Gamma^3_{21}(E_1 - E_3) , \quad (34)$$

$$\partial_1(E_3 - E_2) = \Gamma^3_{31}(E_1 - E_3) - \Gamma^2_{21}(E_1 - E_2) , \quad (35)$$

$$\partial_2(E_3 - E_1) = \Gamma^3_{32}(E_2 - E_3) - \Gamma^1_{12}(E_2 - E_1) , \quad (36)$$

$$\partial_3(E_2 - E_1) = \Gamma^2_{23}(E_3 - E_2) - \Gamma^1_{13}(E_3 - E_1) . \quad (37)$$

5 Time propagation of constraints

For the propagation equations to be integrable with the chosen restrictions on the kinematic variables ($w_{\mu\nu} = 0 = H_{\mu\nu}$), the constraint equations (20-37) must be preserved during the time development of the system. We focus here on the H constraint (32-37); a similar analysis is also valid for constraints (20-31).

The time propagation of the \dot{H} constraint in covariant form (10) gives

$$0 = h^{(i}{}_n \eta^{j)klm} \left[(\dot{E}^n{}_l)_{;m} - E^n{}_{l;p} u^p{}_{;m} - R^n{}_{mpq} E^q{}_l u^p + R^q{}_{mpl} E^n{}_q u^p \right]. \quad (38)$$

If we convert the new constraint (38) to the tetrad form we obtain the following. For $\mu = \nu$, we find

$$0 = \Gamma^1{}_{23}(E_3 - E_1)(\sigma_2 - \sigma_3), \quad (39)$$

$$0 = \Gamma^2{}_{31}(E_1 - E_2)(\sigma_1 - \sigma_3), \quad (40)$$

$$0 = \Gamma^3{}_{12}(E_2 - E_3)(\sigma_1 - \sigma_2), \quad (41)$$

which can be written equivalently as:

$$0 = \Gamma^1{}_{32}(E_2 - E_1)(\sigma_2 - \sigma_3), \quad (42)$$

$$0 = \Gamma^2{}_{13}(E_3 - E_2)(\sigma_1 - \sigma_3), \quad (43)$$

$$0 = \Gamma^3{}_{21}(E_1 - E_3)(\sigma_1 - \sigma_2). \quad (44)$$

For $\mu \neq \nu$, we get

$$\begin{aligned} 0 &= (E_2 - E_3)\partial_1\sigma_3 + (\sigma_2 - \sigma_3)\partial_1E_2 + \frac{1}{3}\Gamma^3{}_{31}(\sigma_1 - \sigma_3)(5E_1 + 4E_3) \\ &- \frac{1}{3}\Gamma^2{}_{21}(E_1 - E_2)(5\sigma_1 + 4\sigma_2), \end{aligned} \quad (45)$$

$$\begin{aligned} 0 &= (E_3 - E_1)\partial_2\sigma_1 + (\sigma_3 - \sigma_1)\partial_2E_3 + \frac{1}{3}\Gamma^1{}_{12}(\sigma_2 - \sigma_1)(5E_2 + 4E_1) \\ &- \frac{1}{3}\Gamma^3{}_{32}(E_2 - E_3)(5\sigma_2 + 4\sigma_3), \end{aligned} \quad (46)$$

$$\begin{aligned} 0 &= (E_1 - E_2)\partial_3\sigma_2 + (\sigma_1 - \sigma_2)\partial_3E_1 + \frac{1}{3}\Gamma^2{}_{23}(\sigma_3 - \sigma_2)(5E_3 + 4E_2) \\ &- \frac{1}{3}\Gamma^1{}_{13}(E_3 - E_1)(5\sigma_3 + 4\sigma_1). \end{aligned} \quad (47)$$

(It is not obvious how the curvature terms in (38) cancel in the transition to the tetrad forms; this is shown in Appendix B).

6 Specializations

6.1 Type O

For this class, $E_\nu = 0$, all constraint equations are trivially satisfied except for the Friedmann equation which controls the dynamics, and the evolution is that of the FRW models.

6.2 Type D

Without loss of generality we set $E_1 = E_2 = E \neq E_3$. The following tetrad properties are valid:

1. From equations (37) and (32,33) we obtain:

$$\Gamma^1_{13} = \Gamma^2_{23}, \quad (48)$$

$$\Gamma^1_{23} = \Gamma^2_{13} = 0. \quad (49)$$

2. From either (26) or (27) we write:

$$(\sigma_2 - \sigma_1)\Gamma^1_{32} = 0, \quad (50)$$

which has the following two subcases.

- (a) For $\sigma_2 = \sigma_1 = \sigma \neq \sigma_3$ the tetrad is free by a rotation in the e_1, e_2 plane. As pointed out earlier, a rotation in that plane can be performed so that equation (16) remains valid. This leaves the tetrad arbitrary by an initial rotation in a surface $x^0 = \text{const}$. To determine the tetrad completely we perform a further rotation of e_1, e_2 which preserves (19), where the value of $\partial\phi/\partial x^3$ is determined from the requirement that $\Gamma^1_{32} = 0$ in a surface $x^0 = \text{const}$. From the Jacobi identities (87-89) in appendix it follows that

$$\Gamma^1_{32} = 0 \quad (51)$$

everywhere. The tetrad vectors can hence be chosen to be hypersurface orthogonal. This result was also obtained in [1]. To complete the consistency analysis we use the above properties as follows:

- i. Constraints (29,30) and (35, 36) are written respectively as

$$\partial_1\sigma = -\sigma\Gamma^3_{31}; \quad \partial_2\sigma = -\sigma\Gamma^3_{32}, \quad (52)$$

$$\partial_1E = -E\Gamma^3_{31}; \quad \partial_2E = -E\Gamma^3_{32}, \quad (53)$$

with constraints (31) and (37) identically satisfied due to equation (48).

ii. First we recall that the time propagation of constraints (26-31) are identically satisfied. Also the time propagation equations (39-41) and (47) are identically satisfied in this class. The identity $0 = 0$ from (47) follows as expected from the $0 = 0$ in (37). On the other hand time propagation equations (45,46) take the forms;

$$0 = -2E\partial_1\sigma + \sigma\partial_1E - \sigma E\Gamma^3_{31} , \quad (54)$$

$$0 = E\partial_2\sigma + 2\sigma\partial_2E + \sigma E\Gamma^3_{32} \quad (55)$$

which are identically satisfied on using (52) and (53).

For this class integrability conditions are consistently satisfied.

(b) For $\sigma_2 \neq \sigma_1$:

We first note that if $\sigma_2 \neq \sigma_1 = \sigma_3$ then from (11) we get $E_1 = E_3$. Similarly $\sigma_1 \neq \sigma_2 = \sigma_3$ implies $E_2 = E_3$. Both cases falls off the specified Type D class. So for non-vanishing shear the eigenvalues are distinct. Furthermore from the time propagation equation (11) we get

$$0 = \dot{E}_2 - \dot{E}_1 = (\sigma_2 - \sigma_1)(3E - \frac{1}{2}\rho) \quad (56)$$

from which it follows that

$$E = \frac{1}{6}\rho . \quad (57)$$

Taking the time derivative of (57) gives

$$E(\sigma_1 + \sigma_2) = 0 , \quad (58)$$

from we get $E = 0$ (iff $\sigma = 0$) or

$$(\sigma_1 + \sigma_2) = 0 . \quad (59)$$

By a series of three further time derivative of (59) one may show that the eigenvalues E_i of the Weyl tensor vanish (i.e., it is type O rather than type D), and hence is a FRW solution.

This proves that for irrotational dust with a purely electric type Weyl tensor that is degenerate the shear is also degenerate in the same plane; furthermore, the integrability conditions are satisfied.

6.3 Type 1

In this case, $E_1 \neq E_2 \neq E_3 \neq E_1$. We deduce the following properties;

1. From the propagation equations (11) it follows that the shear eigenvalues are also distinct. For if say $\sigma_1 = \sigma_2$ then from (11) we get $E_1 = E_2$ which contradicts the requirements of this class.
2. The tetrad vectors are uniquely determined.
3. From (12) we obtain $\Gamma^\nu_{0\mu} = 0$ for $\mu \neq \nu$ and hence the spatial tetrad vectors are Fermi propagated.
4. From the new constraint (39-41), that is the time development of the constraint equations (32-34), we note that the spatial tetrad vectors are hypersurface orthogonal [1] i.e., $\Gamma^1_{23} = \Gamma^2_{31} = \Gamma^3_{12} = 0$.

For this class further analysis of constraints (45-47) is performed below.

6.3.1 Equivalent Constraints

We now concentrate on the new constraint (46). If we write the second term on the right hand side of constraint (46) as

$$(\sigma_3 - \sigma_1)\partial_2 E_3 = (\sigma_3 - \sigma_1)\partial_2(E_3 - E_1) + (\sigma_3 - \sigma_1)\partial_2 E_1 \quad (60)$$

and use (36) we obtain the constraint

$$\begin{aligned} & (E_3 - E_1)\partial_2 \sigma_1 + (\sigma_3 - \sigma_1)\partial_2 E_1 \\ &= \frac{2}{3}\Gamma^3_{32}(\sigma_2 - \sigma_3)(E_2 - E_3) - \frac{2}{3}\Gamma^1_{12}(\sigma_2 - \sigma_1)(E_2 - E_1) \\ &- \Gamma^1_{12}[(\sigma_3 - \sigma_1)(E_2 - E_1) + (\sigma_2 - \sigma_1)(E_3 - E_1)] . \end{aligned} \quad (61)$$

Similar manipulations of the first term on the r.h.s. of (46) yields the following constraint:

$$\begin{aligned} & (E_3 - E_1)\partial_2 \sigma_3 + (\sigma_3 - \sigma_1)\partial_2 E_3 \\ &= \frac{2}{3}\Gamma^3_{32}(\sigma_2 - \sigma_3)(E_2 - E_3) - \frac{2}{3}\Gamma^1_{12}(\sigma_2 - \sigma_1)(E_2 - E_1) \\ &- \Gamma^3_{32}[(\sigma_3 - \sigma_1)(E_2 - E_3) + (\sigma_2 - \sigma_3)(E_3 - E_1)] . \end{aligned} \quad (62)$$

Each of equations (61,62) is equivalent to the constraint (46), if we assume the remaining constraint equations to be satisfied.

6.3.2 Consistency test

To test consistency of the two constraints (61,62) we simplify as follows: Let

$$X_1 = \partial_2 \sigma_1; \quad X_3 = \partial_2 \sigma_3; \quad Y_1 = \partial_2 E_1; \quad Y_3 = \partial_2 E_3, \quad (63)$$

$$a = (E_3 - E_1); \quad b = (\sigma_3 - \sigma_1). \quad (64)$$

So now (61,62) become

$$aX_1 + bY_1 = R_1, \quad (65)$$

$$aX_3 + bY_3 = R_3, \quad (66)$$

with R_1 and R_3 given by the respective r.h.s. Two further constraints are obtained from (65,66) by subtracting and adding respectively i.e.,

$$a(X_3 - X_1) + b(Y_3 - Y_1) = R_3 - R_1, \quad (67)$$

$$a(X_3 + X_1) + b(Y_3 + Y_1) = R_3 + R_1. \quad (68)$$

Constraint (67,68) are due to the time propagation of the constraint (36).

Next we modify the original constraints as follows: multiplying (30) by $a = (E_3 - E_1)$ and (36) by $b = (\sigma_3 - \sigma_1)$ gives:

$$aX_3 = aX_1 + L_1, \quad (69)$$

$$bY_3 = bY_1 + L_3, \quad (70)$$

where now $L_1 = a \times (\text{rhs of (30)})$ and $L_3 = b \times (\text{rhs of (36)})$. Taking a hint from (67, 68) we use constraint (69,70) to formulate the following two further constraints as;

$$a(X_3 - X_1) + b(Y_3 - Y_1) = L_1 + L_3, \quad (71)$$

$$a(X_3 + X_1) + b(Y_3 + Y_1) = 2aX_1 + 2bY_1 + L_1 + L_3, \quad (72)$$

which have not been time propagated. If we substitute (67,68) into (69), (70) respectively we obtain:

$$L_1 + L_3 = R_3 - R_1 \quad (73)$$

and

$$2aX_1 + 2bY_1 + L_1 + L_3 = R_3 + R_1 \quad (74)$$

which is a further set of constraints that are true if and only if (61,62) hold. However equation (74) may be reduced to equation (73) on using (65). This reduction is expected since the two equations (67,68) result from one constraint, namely (46). Thus all we did here was to split one equation (46) into two equivalent new constraints (69,70). And by inserting original constraints relevantly modified as (71,72) into the (67,68) and simplifying we obtain one new constraint.

More explicitly the new constraint (73) can be written as

$$\begin{aligned}
& (E_3 - E_1) \left[\Gamma^3_{32}(\sigma_2 - \sigma_3) - \Gamma^1_{12}(\sigma_2 - \sigma_1) \right] \\
& + (\sigma_3 - \sigma_1) \left[\Gamma^3_{32}(E_2 - E_3) - \Gamma^1_{12}(E_2 - E_1) \right] \\
& = \frac{2}{3} \Gamma^3_{32}(\sigma_2 - \sigma_3)(E_2 - E_3) - \frac{2}{3} \Gamma^1_{12}(\sigma_2 - \sigma_1)(E_2 - E_1) \\
& - \Gamma^1_{12} [(\sigma_3 - \sigma_1)(E_2 - E_1) + (\sigma_2 - \sigma_1)(E_3 - E_1)] \\
& - \frac{2}{3} \Gamma^3_{32}(\sigma_2 - \sigma_3)(E_2 - E_3) + \frac{2}{3} \Gamma^1_{12}(\sigma_2 - \sigma_1)(E_2 - E_1) \\
& + \Gamma^3_{32} [(\sigma_3 - \sigma_1)(E_2 - E_3) + (\sigma_2 - \sigma_3)(E_3 - E_1)] \tag{75}
\end{aligned}$$

and reduces easily to the identity $0 = 0$; which thus shows that (46), the basis on which these equations were derived, must be identically true (in virtue of all the other constraints that hold). A similar approach for the constraints (45) and (47) yields similar identities $0 = 0$. And hence the set of integrability conditions (45-47) are also satisfied for Type I fields.

7 Coordinates

The vectors of the eigen-tetrad are hypersurface orthogonal for both type D and type I spacetimes. So now using the tetrad-coordinate relations (14,15) there exists a coordinate system in which line element has diagonal form;

$$ds^2 = -V^2 dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 \tag{76}$$

where A, B, C, V are functions of the spacetime variables (Barnes and Rowlinson, [1]). For type D spacetimes, a suitable rescaling of the t coordinate to set $V = 1$ and use of (48) and (52) gives

$$ds^2 = -dt^2 + A^2(x^i)(dx^2 + k(x, y)^2 dy^2) + \sigma^{-2}(x^j)g^2(z, t)dz^2 \tag{77}$$

where $\sigma^2(x^i)$ is the magnitude of the shear; this has the form assumed by Szekeres [5].

8 Conclusion

The result that the tetrad vectors are hypersurface orthogonal appears as an integrability condition from the \dot{H} constraint with $\mu = \nu$, and for the given tetrad choice this condition is satisfied. This has been emphasized in [1]. We have managed to show here that the remaining \dot{H} constraints for $\mu \neq \nu$ are also satisfied without imposing any further geometric requirement on the tetrad. We have also checked that the time derivatives of all the other constraints are identically satisfied in view of the set of constraint equations that hold under our given conditions.

Thus all the constraints resulting from the vanishing of H_{ab} can be consistently satisfied in solutions with $p = 0 = \omega_{ab}$. More specifically, $H_{ab} = 0$ is equivalent to (6); the time derivative of this equation gives (10); if both of these equations are satisfied, there are no further consistency conditions to be satisfied resulting from $H_{ab} = 0$ or higher time derivatives.

The latter result was assumed to be valid in the papers by Mataresse *et al* [6] and Bruni *et al* [2], where on neglecting the effect of surrounding matter, as carried by H_{ab} (in the absence of a pressure gradient), the model evolved as a silent universe.

We can look at this from the viewpoint of initial data: for initial conditions that satisfy the constraint equation $H_{ab} = 0$ (that is, (6)) at some initial time $t = t_0$, together with the constraint equation $\dot{H}_{ab} = 0$ (that is, (10)) at that time, then for later times (within the Cauchy development of this initial data) the magnetic part of the Weyl tensor will remain zero. Thus vanishing of H_{ab} is a consistent condition in an open domain. It is a dynamical restriction which will remain valid through the evolution of the system; thus for example the collapse of fluid elements to a curvature singularity in the form of a Zeldovich pancake can be consistently represented by such solutions.

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A Jacobi Identities

Following [3], the commutators $[e_a, e_b]$ can be written as:

$$[e_a, e_b] = \gamma^c_{ab} e_c, \quad a, b, c = 0 \dots 3, \quad (78)$$

where the structure constants γ^c_{ab} are related to the Ricci coefficients Γ^c_{ab} by:

$$\gamma^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba} \quad (79)$$

and inversely by:

$$\Gamma_{abc} = \frac{1}{2}(\gamma_{abc} + \gamma_{cab} - \gamma_{bca}). \quad (80)$$

The Jacobi identity

$$[e_b, [e_c, e_d]] + [e_d, [e_b, e_c]] + [e_c, [e_d, e_b]] = 0 \quad (81)$$

can then be written in terms of γ^c_{ab} as:

$$\left(\begin{array}{c} f \\ bcd \end{array} \right) : \quad \partial_{[d} \gamma^f_{cb]} + \gamma^g_{[dc} \gamma^f_{b]g} = 0. \quad (82)$$

That is:

$$\left(\begin{array}{c} 1 \\ 123 \end{array} \right) : \quad \partial_1 \gamma^1_{32} - \partial_2 \gamma^1_{31} + \partial_3 \gamma^1_{21} = -\gamma^2_{32} \gamma^1_{21} - \gamma^3_{23} \gamma^1_{31} + \gamma^1_{32} (\gamma^2_{12} - \gamma^3_{13}), \quad (83)$$

$$\left(\begin{array}{c} 2 \\ 123 \end{array} \right) : \quad -\partial_1 \gamma^2_{32} + \partial_2 \gamma^2_{13} - \partial_3 \gamma^2_{12} = -\gamma^1_{31} \gamma^2_{12} + \gamma^3_{13} \gamma^2_{32} + \gamma^2_{13} (\gamma^3_{23} + \gamma^1_{21}), \quad (84)$$

$$\begin{pmatrix} 3 \\ 123 \end{pmatrix} : -\partial_1 \gamma^3_{23} + \partial_2 \gamma^3_{13} - \partial_3 \gamma^3_{12} = \gamma^1_{21} \gamma^3_{13} - \gamma^2_{12} \gamma^3_{23} - \gamma^3_{12} (\gamma^1_{31} + \gamma^2_{32}) , \quad (85)$$

$$\begin{pmatrix} \mu \\ 0\mu\nu \end{pmatrix} : \partial_0 \gamma^\mu_{\nu\mu} = -\partial_\nu \theta_\mu - \theta_\nu \gamma^\mu_{\nu\mu} , \quad (86)$$

$$\begin{pmatrix} 1 \\ 023 \end{pmatrix} : \partial_0 \gamma^1_{32} = \gamma^1_{32} (\theta_1 - \theta_2 - \theta_3) , \quad (87)$$

$$\begin{pmatrix} 2 \\ 031 \end{pmatrix} : \partial_0 \gamma^2_{13} = \gamma^2_{13} (-\theta_1 + \theta_2 - \theta_3) , \quad (88)$$

$$\begin{pmatrix} 3 \\ 012 \end{pmatrix} : \partial_0 \gamma^3_{12} = \gamma^3_{12} (-\theta_1 - \theta_2 + \theta_3) . \quad (89)$$

B Sample Calculations

We consider here the time propagation of the \dot{H} constraint. Starting from the covariant form [see equation (10)]

$$\dot{H} : \quad h^{(i}{}_n \eta^{j)klm} u_k E^n_{l;m} = 0 \quad (90)$$

and using

$$(\dot{E}^n_{l;m}) = (\dot{E}^n_l)_{;m} - E^n_{;p} u^p_{;m} - R^n_{mpq} E^q_l u^p + R^q_{mpl} E^n_q u^p , \quad (91)$$

we obtain as the time derivative of (90),

$$0 = h^{(i}{}_n \eta^{j)klm} u_k \left[(\dot{E}^n_l)_{;m} - E^n_{l;p} u^p_{;m} - R^n_{mpq} E^q_l u^p + R^q_{mpl} E^n_q u^p \right] . \quad (92)$$

The two terms involving the Riemann tensor R^a_{bcd} may be shown to be zero as follows:

$$\begin{aligned} & h^{(i}{}_n \eta^{j)klm} u_k [R^q_{mpl} E^n_q u^p - R^n_{mpq} E^q_l u^p] \\ &= \frac{1}{2} \eta^{jklm} u_k [R^q_{mpl} E^i_q u^p - R^i_{mpq} E^q_l u^p] \\ & \quad + \frac{1}{2} \eta^{iklm} u_k [R^q_{mpl} E^j_q u^p - R^j_{mpq} E^q_l u^p] \\ &= \frac{1}{2} (Term\ 1 + Term\ 2) . \end{aligned} \quad (93)$$

where

$$\begin{aligned} Term\ 1 &= \frac{1}{2}\eta^{jklm}u_k [R^q_{mpl}E^i_q u^p - R^i_{mpq}E^q_l u^p] \\ Term\ 2 &= \frac{1}{2}\eta^{ijklm}u_k [R^q_{mpl}E^j_q u^p - R^j_{mpq}E^q_l u^p] . \end{aligned} \quad (94)$$

Consider now

$$\begin{aligned} Term\ 1 &= \frac{1}{2}\eta^{jklm}u_k [R^q_{mpl}E^i_q u^p - R^i_{mpq}E^q_l u^p] \\ &= T1 + T2 \end{aligned} \quad (95)$$

where we set

$$T1 = \eta^{jklm}E^q_l R^i_{mpq}u_k u^p ; \quad (96)$$

$$T2 = \eta^{ijklm}E^i_q R^q_{mpl}u_k u^p . \quad (97)$$

We write the Riemann tensor R_{smpl} in the terms of the Weyl tensor as

$$\begin{aligned} R_{smpl} &= C_{smpl} + \frac{1}{2}(g_{sp}R_{lm} + g_{sl}R_{pm} - g_{mp}R_{sl} + g_{ml}R_{sp}) \\ &\quad - \frac{R}{6}(g_{sp}g_{lm} - g_{sl}g_{pm}) \end{aligned} \quad (98)$$

where

$$C_{smpl} \equiv (\eta_{smab}\eta_{plcd} + g_{smab}g_{plcd})u^a u^c E^{bd}, \quad (99)$$

and

$$g_{smab} \equiv g_{sa}g_{mb} - g_{sb}g_{ma}. \quad (100)$$

Calculation of T1: Using (98) in (96) we get

$$\begin{aligned} T1 &= \eta^{jklm}E^q_l R^i_{mpq}u^p u_k \\ &= \eta^{jklm}E^q_l u^p u_k [C^i_{mpq} + \frac{1}{2}(g^i_p R_{qm} + g^i_q R_{pm} - g_{mp}R^i_q + g_{mq}R^i_p) \\ &\quad - \frac{R}{6}(g^i_p g_{qm} - g^i_q g_{pm})] \\ &= T1a + T1b \end{aligned} \quad (101)$$

where

$$\begin{aligned}
T1a &= \eta^{jklm} E^q_l u^p u_k C^i_{mpq} \\
&= E^q_l E^{bd} u^a u^c u^p u_k [\eta^{jklm} \eta^i_{mab} \eta_{pqcd} + \eta^{jklm} (g^i_a g_{mb} - g^i_b g_{ma}) g_{pqcd}] \\
&= E^q_l E^{bd} \eta^{jklm} \eta^i_{mab} \eta_{pqcd} u^a u^c u^p u_k + E^q_l E^d_m \eta^{jklm} u^i u^c u^p u_k g_{pqcd} \\
&\quad + E^q_l E^{id} \eta^{jklm} u_m u^c u^p u_k g_{pqcd} \\
&= E^q_l E^d_m \eta^{jklm} u^i u^c u^p u_k (g_{pc} g_{qd} - g_{pd} g_{qc}) \\
&= E_{ld} E^d_m \eta^{jklm} u^i u_k u_p u^p - E_{lc} E_{mp} \eta^{jklm} u^i u^c u^p u_k \\
&= 0 \quad (\text{Symmetry properties})
\end{aligned} \tag{102}$$

and

$$\begin{aligned}
T1b &= \eta^{jklm} E^q_l u^p u_k [\tfrac{1}{2} (g^i_p R_{qm} + g^i_q R_{pm} - g_{mp} R^i_q + g_{mq} R^i_p) \\
&\quad - \tfrac{R}{6} (g^i_p g_{qm} - g^i_q g_{pm})] \\
&= \eta^{jklm} u_k [\tfrac{1}{2} (E^q_l R_{qm} u^i + E^i_l R_{pm} u^p) - \tfrac{1}{2} (E^q_l R^i_q u_m - E_{ml} R^i_p u^p) \\
&\quad - \tfrac{R}{6} (E_{ml} u^i - E^i_l u_m)].
\end{aligned} \tag{103}$$

For dust

$$R_{pm} = \tfrac{1}{2} \rho (h_{pm} + u_p u_m). \tag{104}$$

and hence (103) becomes

$$\begin{aligned}
T1b &= -\tfrac{1}{2} \rho E^i_l \eta^{jklm} u_m u_k - \tfrac{R}{6} (E_{ml} u^i - E^i_l u_m) \eta^{jklm} u_k \\
&= 0 \quad (\text{Symmetry properties}).
\end{aligned} \tag{105}$$

So now $T1 = T1a + T1b = 0$.

Calculation of T2: Using (98) in (97) we get

$$\begin{aligned}
T2 &= \eta^{jklm} E^i_q R^q_{mpl} u_k u^p \\
&= \eta^{jklm} E^i_q u_k u^p [C^q_{mpl} + \tfrac{1}{2} (g^q_p R_{lm} + g^q_l R_{pm}) - \tfrac{1}{2} (g_{mp} R^q_l - g_{ml} R^q_p) \\
&\quad - \tfrac{1}{6} (g^q_p g_{lm} - g^q_l g_{pm})] \\
&= T2a + T2b
\end{aligned} \tag{106}$$

and as in (101) we set

$$\begin{aligned}
T2a &= \eta^{jklm} E^i_q u_k u^p C^q_{mpl} \\
&= \eta^{jklm} E^i_q E^{bd} u^a u^c u_k u^p [\eta^q_{mab} \eta_{plcd} + (g^q_a g_{mb} - g^q_b g_{ma}) g_{plcd}] \\
&= \eta^{jklm} \eta^q_{mab} \eta_{plcd} E^i_q E^{bd} u^a u^c u_k u^p - \eta^{jklm} E^i_b E^{bd} u^c u^p u_m u_k (g_{pc} g_{ld} - g_{pd} g_{lc}) \\
&= 0 \quad (\text{Symmetry properties.})
\end{aligned} \tag{107}$$

and

$$\begin{aligned}
T2b &= \eta^{jklm} E^i_q u_k u^p [\tfrac{1}{2}(g^q_p R_{lm} + g^q_l R_{pm}) - \tfrac{1}{2}(g_{mp} R^q_l - g_{ml} R^q_p) \\
&\quad - \tfrac{R}{6}(g^q_p g_{lm} - g^q_l g_{pm})] \\
&= 0 \quad (\text{Symmetry properties.})
\end{aligned} \tag{108}$$

Here also $T2 = T2a + T2b = 0$ and hence from (95)

$$Term\ 1 = 0. \tag{109}$$

A similar program of calculations with i and j interchanged gives $Term\ 2 = 0$ and the equation (92) takes the final form

$$0 = h^{(i}_n \eta^{j)klm} u_k \left[(\dot{E}^n_l)_{;m} - E^n_{l;p} u^p_{;m} \right]. \tag{110}$$

In tetrad form (110) becomes

$$\begin{aligned}
0 &= h^{(\mu}_\tau \eta^{\nu)\kappa\alpha\beta} u_\kappa \left[\partial_\beta \dot{E}^\tau_\alpha - \theta^p_\beta \partial_p E^\tau_{\alpha;p} + \Gamma^\tau_{\beta\epsilon} \dot{E}^\epsilon_\alpha - \Gamma^\epsilon_{\beta\alpha} \dot{E}^\tau_\epsilon \right. \\
&\quad \left. - \theta^p_\beta (\Gamma^\tau_{p\epsilon} E^\epsilon_\alpha - \Gamma^\epsilon_{p\alpha} E^\tau_\epsilon) \right].
\end{aligned} \tag{111}$$

Setting $\mu = \nu$ in (111) yields:

$$0 = \eta^{\mu\kappa\alpha\beta} u_\kappa \Gamma^\mu_{\beta\alpha} \left[(\dot{E}_\alpha - \dot{E}_\mu) - \theta_\beta (E_\alpha - E_\mu) \right]. \tag{112}$$

If we substitute the propagation equation (11) in (112) and consider the values of $\mu = 1, 2, 3$ we obtain the equations

$$0 = \Gamma^1_{23} (E_3 - E_1) (\sigma_2 - \sigma_3), \tag{113}$$

$$0 = \Gamma^2_{31} (E_1 - E_2) (\sigma_1 - \sigma_3), \tag{114}$$

$$0 = \Gamma^3_{12} (E_2 - E_3) (\sigma_1 - \sigma_2). \tag{115}$$

which are the constraints (39-41).

For values $\mu \neq \nu$, say $\mu = 1, \nu = 3$, (111) becomes

$$\begin{aligned} 0 &= \partial_2(\dot{E}_3 - \dot{E}_1) - \theta_2 \partial_2(E_3 - E_1) + \Gamma^1_{12} [(\dot{E}_2 - \dot{E}_1) - \theta_1(E_2 - E_1)] \\ &\quad - \Gamma^3_{32} [(\dot{E}_2 - \dot{E}_3) - \theta_3(E_2 - E_3)] \end{aligned} \quad (116)$$

If we substitute the propagation equation (11) into (116) and use the original constraint equations repeatedly we obtain

$$\begin{aligned} 0 &= (E_3 - E_1) \partial_2 \sigma_1 + (\sigma_3 - \sigma_1) \partial_2 E_3 + \frac{1}{3} \Gamma^1_{12} (\sigma_2 - \sigma_1) (5E_2 + 4E_1) \\ &\quad - \frac{1}{3} \Gamma^3_{32} (E_2 - E_3) (5\sigma_2 + 4\sigma_3); \end{aligned} \quad (117)$$

The remaining set of $\mu = 2, \nu = 3$ and $\mu = 1, \nu = 2$ yield

$$\begin{aligned} 0 &= (E_2 - E_3) \partial_1 \sigma_3 + (\sigma_2 - \sigma_3) \partial_1 E_2 + \frac{1}{3} \Gamma^3_{31} (\sigma_1 - \sigma_3) (5E_1 + 4E_3) \\ &\quad - \frac{1}{3} \Gamma^2_{21} (E_1 - E_2) (5\sigma_1 + 4\sigma_2); \end{aligned} \quad (118)$$

and

$$\begin{aligned} 0 &= (E_1 - E_2) \partial_3 \sigma_2 + (\sigma_1 - \sigma_2) \partial_3 E_1 + \frac{1}{3} \Gamma^2_{23} (\sigma_3 - \sigma_2) (5E_3 + 4E_2) \\ &\quad - \frac{1}{3} \Gamma^1_{13} (E_3 - E_1) (5\sigma_3 + 4\sigma_1) \end{aligned} \quad (119)$$

respectively, which are the constraints (45-47).

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